# DELOCALIZED EQUIVARIANT COHOMOLOGY AND RESOLUTION

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Abstract. A refined form of the 'Folk Theorem' that a smooth action by a compact Lie group can be (canonically) resolved, by iterated blow up, to have unique isotropy type was established in [1] in the context of manifolds with corners; the canonical construction induces fibrations on the boundary faces of the resolution resulting in an 'equivariant resolution structure'. Here, equivariant K-theory and the Cartan model for equivariant cohomology are tracked under the resolution procedure as is the delocalized equivariant cohomology of Baum, Brylinski and MacPherson. This leads to resolved models for each of these cohomology theories, in terms of relative objects over the resolution structure and hence to reduced models as flat-twisted relative objects over the resolution of the quotient. An explicit equivariant Chern character is then constructed, essentially as in the non-equivariant case, over the resolution of the quotient.

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## 1. Introduction

In a recent paper [1], the authors established a form of the 'Folk Theorem' that the smooth action of a general compact Lie group, $G$, on a compact manifold, $X$, can be resolved to have components with unique isotropy type. In this paper, cohomological consequences of this construction are derived. In particular, directly on the resolution, a 'delocalized' equivariant cohomology is defined, and shown to reduce to the cohomology of Baum, Brylinski and MacPherson [7] in the Abelian case. The equivariant Chern character is then obtained from the usual Chern character by twisting with flat coefficients and is shown to induce an isomorphism between delocalized equivariant cohomology and equivariant K-theory with complex coefficients.

A smooth action is 'resolved' if it has a unique isotropy type, i.e., if all of the stabilizer groups are conjugate to each other, in which case Borel showed that the orbit space $G\backslash X$ is a smooth manifold. If a $G$-action on $X$ is not resolved then the construction in [1] produces a manifold with corners $Y(X)$, the canonical resolution of $X$, with a resolved $G$-action and an equivariant 'blow-down map', $Y(X) \xrightarrow{\beta} X$. Each isotropy type of $X$,

$$X^{[K]} = \{\zeta \in X : \text{ stabilizer of } \zeta \text{ is conjugate to } K\}$$

is associated with a boundary hypersurface, $M_{[K]}$, of $Y(X)$. Indeed, $X^{[K]}$ is itself a manifold with a smooth $G$-action, and $M_{[K]}$ is the total space of an equivariant fibration over the canonical resolution of the closure of $X^{[K]}$. The different isotropy types of $X$ determine a stratification of $X$ and the inclusion relation between the strata corresponds to the intersection relation between the corresponding boundary faces of the canonical resolution of $X$. The result is that $Y(X)$ has an 'equivariant resolution structure', which encapsulates the relations between the different isotropy types of $X$.

As the quotient of the resolved group action is smooth, the canonical resolution of $X$ induces a resolution, $Z(X)$, of the quotient $G\backslash X$ in a similar form, as a compact manifold with corners with fibrations of its boundary hypersurfaces over the quotients of the resolutions of the isotropy types.

The Cartan model for the equivariant cohomology, $\mathrm{H}^*_G(X)$, can be lifted to the resolution and then projected to the quotient. In the free case, a theorem of Borel identifies this localized equivariant cohomology with the cohomology of the quotient. In the case of a group action with unique isotropy type we show that the equivariant cohomology reduces to the cohomology over the quotient with coefficients in a flat bundle of algebras, which we call the *Borel bundle*, modeled on the invariant polynomials on the Lie algebra of the isotropy group – or equivalently the $G$-invariant and symmetric part of the total tensor product of the dual. In the general case the equivariant cohomology is identified with the relative cohomology, with respect to the resolution structure, twisted at each level by this flat coefficient bundle; the naturality of the bundle ensures that there are consistency maps under the boundary fibrations induced on the twisted forms. Thus the Borel bundle represents the only equivariant information over the resolution of the quotient needed to recover the equivariant cohomology. In this construction we adapt Cartan's form of the isomorphism in the free case as presented by Guillemin and Sternberg [12] to the case of a fixed isotropy group.

Using the approach through $G$-equivariant bundles as discussed by Atiyah and Segal [6, 16] we give a similar lift of the equivariant K-theory, $\mathrm{K}^*_G(M)$, to the resolution of the action and then project to the resolution of the quotient. This results in a closely analogous reduced model for equivariant K-theory where the Borel bundles are replaced by what we term the 'representation bundles', which are flat bundles of rings modelled on the representation ring of the isotropy group over each resolved isotropy type. Cartan's form of the Borel-Weil construction gives a map back to the Borel bundle.

The representation bundles over the resolution structure amount to a resolution (in the Abelian case where it was initially defined) of the sheaf used in the construction of the delocalized equivariant cohomology of Baum, Brylinski and MacPherson ([7], see also [9]). The close parallel between the reduced models for equivariant K-theory and equivariant cohomology allow us to introduce, directly on the resolution, a *delocalized* deRham cohomology $\mathrm{H}^*_{G,\mathrm{dl}}(X)$ generalizing the construction in [7] to the case a general compact group action. As expected and as in the Abelian case, the Chern character gives an isomorphism,

$$\mathrm{Ch}_G : \mathrm{K}_G(M) \otimes_{\mathbb{Z}} \mathbb{C} \longrightarrow \mathrm{H}^{\mathrm{even}}_{G,\mathrm{dl}}(X) \tag{1}$$

from equivariant K-theory with complex coefficients. These results are also related to the work of Rosu, [15], and earlier work of Illman, [13], in the topological setting and likely carry over to other cohomology theories.

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## 2. Group actions

We review the basic definitions and constructions we will need below. For details we refer to [11] and [1].

Let $X$ be a closed manifold and $G$ a compact Lie group. By a smooth action of $G$ on $X$ we mean a smooth homomorphism from $G$ into the diffeomorphisms of $X$,

$$A : G \longrightarrow \mathrm{Dfo}(X).$$

If $g \in G$, we usually denote $A(g)$ by

$$g : X \ni \zeta \longmapsto g\zeta \in X.$$

The differential of the action of $G$ on itself by conjugation is the adjoint representation of $G$ on its Lie algebra, $\mathfrak{g}$,

$$\mathrm{Ad}_G : G \longrightarrow \mathrm{End}(\mathfrak{g}).$$

An inner product on $\mathfrak{g}$ is $\mathrm{Ad}_G$-invariant precisely when the induced metric on $G$ is invariant under left and right translations; by the usual averaging argument, such a metric exists if $G$ is compact.

The action of $G$ on $X$ induces a map of Lie algebras, from the Lie algebra $\mathfrak{g} = T_{\mathrm{Id}}G$ of $G$ into the Lie algebra of vector fields of $X$,

$$\alpha : \mathfrak{g} \longrightarrow \mathcal{C}^\infty(X;TX), \quad \alpha(V)(\zeta) = \partial_t e^{-tV}\zeta\big|_{t=0} \in T_\zeta X$$

where exponentiation in $G$ is done via an $\mathrm{Ad}_G$-invariant metric.

Although such a group action is smooth, by definition, it is in general rather seriously 'non-uniform' in the sense that the orbits, necessarily individually smooth,

change dimension locally. This is encoded in the *isotropy* (also called stabilizer) groups. Namely for $\zeta \in X$ set

$$G_\zeta = \{g \in G : g\zeta = \zeta\}\,. \tag{2.1}$$

This is a Lie subgroup and long-established local regularity theory shows that if $K \subset G$ is a Lie subgroup then

$$X^K = \{\zeta \in X : G_\zeta = K\}, \tag{2.2}$$

is a smooth submanifold. Since

$$G_{g\zeta} = g(G_\zeta)g^{-1}, \tag{2.3}$$

the conjugate of an isotropy group is an isotropy group and if $[K]$ is the class of subgroups conjugate to $K$ then

$$X^{[K]} = \{\zeta \in X : G_\zeta = gKg^{-1} \text{ for some } g \in G\} \tag{2.4}$$

is also a smooth submanifold of $X$, the *isotropy type* of $X$.

The natural partial order on conjugacy classes of subgroups of $G$ corresponding to inclusion

$$[K_1] \leq [K_2] \iff \text{there is } g \in G \text{ s.t. } K_1 \subseteq gK_2g^{-1},$$

induces an order on isotropy types of $X$. It is standard to take the opposite order, thus

$$X^{[K_2]} \leq X^{[K_1]} \iff [K_1] \leq [K_2], \tag{2.5}$$

as larger isotropy groups correspond to smaller submanifolds. The isotropy types, or *isotypes*, give a stratification of $X$. We will denote the set of all conjugacy classes of isotropy groups of the $G$-action on $X$ by $\mathcal{I}$.

## 3. Equivariant cohomology and K-theory

The basic topological invariants of a group action are the equivariant cohomology and K-theory groups. We briefly review the definitions of these theories and refer the reader to [12] and [16] for more information.

We start by recalling Cartan's model for the equivariant cohomology of $X$. An *equivariant differential form* is a polynomial map

$$\omega : \mathfrak{g} \longrightarrow \mathcal{C}^\infty(X, \Lambda^* X)$$

which is invariant in that it intertwines the adjoint actions of $G$ on $\mathfrak{g}$ and the pull-back action

$$g\cdot : \mathcal{C}^\infty(X, \Lambda^* X) \ni \omega \mapsto (g^{-1})^* \omega \in \mathcal{C}^\infty(X, \Lambda^* X).$$

This can also be thought of as a $G$-invariant section of the bundle $S(\mathfrak{g}^*) \otimes \Lambda^*$ where $S(\mathfrak{g}^*)$, the symmetric part of the tensor powers of the dual $\mathfrak{g}^*$, is identified with the ring of polynomials on $\mathfrak{g}$. We denote the set of these smooth equivariant forms by $\mathcal{C}^\infty_G(M; S(\mathfrak{g}^*) \otimes \Lambda^*)$; they form an algebra with respect to the usual wedge product. This algebra is graded by defining

$$\text{degree}(\omega) = \text{differential form degree}(\omega) + 2\,(\text{polynomial degree}(\omega)) \tag{3.1}$$

on homogeneous elements.

The *equivariant differential*

$$\begin{gathered} d_{\mathrm{eq}} : \mathcal{C}^\infty_G(X; S(\mathfrak{g}^*) \otimes \Lambda^*) \longrightarrow \mathcal{C}^\infty_G(X; S(\mathfrak{g}^*) \otimes \Lambda^*), \\ (d_{\mathrm{eq}}\omega)(v) = d(\omega(v)) - \mathfrak{i}_{\alpha(v)}(\omega(v)),\ \forall\ v \in \mathfrak{g}, \end{gathered}$$

then increases the degree by one and satisfies $d_{\mathrm{eq}}^2 = 0$. The resulting cohomology groups are the equivariant cohomology groups and will be denoted $\mathrm{H}^q_G(X)$.

The equivariant cohomology of a point is the space of polynomials on $\mathfrak{g}$ invariant with respect to the adjoint action of $G$, $\mathrm{H}^*_G(\mathrm{pt}) = S(\mathfrak{g}^*)^G$. Thus, if $\dim \mathfrak{g} > 0$, the groups $\mathrm{H}^q_G(\mathrm{pt})$ are non-zero in all even degrees. The equivariant cohomology groups of a space $X$ are always modules over $\mathrm{H}^*_G(\mathrm{pt})$, and are usually non-zero in infinitely many degrees.

We use the model for equivariant K-theory as the Grothendieck group based on equivariant complex vector bundles over the manifold. Thus an (absolute) equivariant K-class on $X$ is fixed by a pair of vector bundles $E^\pm$ each of which has an action of $G$ as bundle isomorphisms covering the action of $G$ on the base:

$$L^\pm_E(g) : g^*E^\pm \longrightarrow E^\pm. \tag{3.2}$$

The equivalence relation fixing a class from such data is stable $G$-equivariant isomorphism so $(E^\pm, L^\pm_E)$ and $(F^\pm, L^\pm_F)$ are equivalent if there exists a $G$-equivariant vector bundle $S$ and an equivariant bundle isomorphism

$$T : E^+ \oplus F^- \oplus S \longrightarrow F^+ \oplus E^- \oplus S. \tag{3.3}$$

Direct sum and tensor product are well-defined among these equivalence classes and the resulting ring is denoted $\mathrm{K}^0_G(X)$. By standard results, this ring is the same if the bundles are required to be smooth or only continuous, and we shall work exclusively with the former.

Odd K-theory is defined as the null space of the pull-back homomorphism corresponding to $\iota : X \hookrightarrow \mathbb{S} \times X$, $\zeta \longmapsto (1, \zeta)$ :

$$\mathrm{K}^1_G(X) \longrightarrow \mathrm{K}^0_G(\mathbb{S} \times X) \xrightarrow{\iota^*} \mathrm{K}^0_G(X) \tag{3.4}$$

where $G$ acts trivially on $\mathbb{S}$. As in the standard case, pull-back under the projection $\mathbb{S} \times X \longrightarrow X$ induces a decomposition

$$\mathrm{K}^0_G(\mathbb{S} \times X) = \mathrm{K}^1_G(X) \oplus \mathrm{K}^0_G(X). \tag{3.5}$$

The equivariant K-theory of a point $\mathrm{K}^*_G(\mathrm{pt}) = \mathrm{K}^0_G(\mathrm{pt})$ is the ring of virtual representations of $G$, $\mathcal{R}(G)$. As a group, $\mathcal{R}(G)$ is the free Abelian group generated by the set of simple $G$-modules, i.e., the irreducible finite dimensional representations of $G$. The equivariant K-theory groups of a space $X$ are always algebras over $\mathrm{K}^0_G(\mathrm{pt})$.

Comparing $\mathrm{H}^*_G(\mathrm{pt})$ with $\mathrm{K}^*_G(\mathrm{pt})$, it is clear that the Atiyah-Hirzebruch theorem cannot extend directly to the equivariant setting. Indeed, from general principles the Chern character is a natural transformation

$$\mathrm{Ch} : \mathrm{K}^0_G(X) \longrightarrow \mathrm{H}^{\mathrm{even}}_G(X), \tag{3.6}$$

and Atiyah and Segal showed [6], [3, Theorem 9.1] that this induces an isomorphism after tensorsing $\mathrm{K}^0_G(X)$ with $\mathbb{C}$ and localizing at the identity element of the group,

$$(\mathrm{K}^0_G(X) \otimes \mathbb{C})^\wedge \xrightarrow{\cong} \mathrm{H}^{\mathrm{even}}_G(X).$$

The localization map $L^\wedge : \mathrm{K}^0_G(X) \otimes \mathbb{C} \longrightarrow (\mathrm{K}^0_G(X) \otimes \mathbb{C})^\wedge$ is usually far from an isomorphism.

This issue was addressed by Baum, Brylinski and MacPherson, [7], who introduced 'delocalized' equivariant cohomology groups, $\mathrm{H}^{\mathrm{even}}_{G,\mathrm{dl}}(X)$, in the case of an Abelian group action, such that the Chern character factors and induces an isomorphism from K-theory with complex coefficients

$$\begin{tikzcd}[column sep=large] \mathrm{Ch} : \mathrm{K}^0_G(X) \arrow[rr] \arrow[dr, "\otimes\mathbb{C}"'] & & \mathrm{H}^{\mathrm{even}}_{G,\mathrm{dl}}(X) \arrow[r, "L^\wedge"] & \mathrm{H}^{\mathrm{even}}_G(X) \\ & \mathrm{K}^0_G(X)\otimes\mathbb{C}. \arrow[ur, "\simeq"'] & & \end{tikzcd} \tag{3.7}$$

One consequence of the lifting of cohomology under resolution procedure discussed here is that the extension of the groups $\mathrm{H}^{\mathrm{even}}_{G,\mathrm{dl}}(X)$ to the non-Abelian case becomes transparent as does the structure of the localization map $L^\wedge$.

## 4. Actions with a unique isotropy type

Since the effect of resolution of the group action is to 'simplify' to the case of a unique istropy type (with the complexity relegated to iteration over the fibrations of the boundary faces) we first discuss the special case in detail. The effect of the working with a manifold with corners (and absolute cohomology and K-theory) is minimal here, and may be mostly be ignored. However recall from [1] that by a manifold with corners we mean a space that that is locally and smoothly modeled on $[0,\infty)^n$, and has boundary hypersurfaces which are embedded. A smooth group action on a manifold with corners is, as part of the definition of smoothness, required to satisfy an extra condition, namely that the orbit of any boundary hypersurface is an embedded submanifold of $X$. This is equivalent to asking that, for any boundary hypersurface $M$ of $X$ and any element $g\in G$,

$$g\cdot M\cap M=\emptyset \text{ or } g\cdot M=M. \tag{4.1}$$

This condition holds on the canonical resolution of a smooth manifold and in any case can always be arranged, see [1].

A compact manifold (possibly with corners) $X$ with a smooth action by a compact Lie group $G$, has a unique isotropy type if all of the isotropy groups of $X$ are conjugate in $G$, so there is a subgroup $K\subset G$ satisfying

$$\zeta\in X \implies \exists\, g\in G \text{ s.t. } gG_\zeta g^{-1}=K$$

or equivalently, $X=X^{[K]}$.

Let $N(K)=\{g\in G : gKg^{-1}=K\}$ denote the normalizer of $K$ in $G$. Then $N(K)$ acts on $X^K$ with $K$ acting trivially, so the action factors through a free action of $N(K)/K$, which is therefore the total space of a principal $N(K)/K$ bundle.

**Proposition 4.1** (Borel)**.** *If $X=X^{[K]}$ the orbit space $G\backslash X$ is smooth with the smooth structure induced by the natural $G$-equivariant diffeomorphism*

$$G\times_{N(K)}X^K \xrightarrow{\cong} X \Longrightarrow G\backslash X\cong N(K)\backslash X^K.$$

*Proof.* The map induced by the action of $G$, $G\times X^K\longrightarrow X$ is surjective, by the assumed uniqueness of the isotropy type and is equivariant under the left action on $G$. The inverse image of a point is an orbit under the free diagonal action of $N(K)$, $(g,\zeta)\longmapsto(gn^{-1},n\zeta)$, with quotient $G\times_{N(K)}X^K$, which leads to the desired isomorphisms. □

The equivariant cohomology of $X$ on the quotient space $Z = G\backslash X$, is conveniently expressed in terms of a family of flat vector bundles as coefficients. Let $\mathfrak{g}_\zeta$ be the Lie algebra of $G_\zeta$.

*Definition* 1. If $X$ is a compact manifold with a smooth $G$-action such that $X = X^{[K]}$ and $j \in \mathbb{N}_0$, let $\widetilde{B}^j \xrightarrow{\pi_j} X$ be the vector bundle over $X$ with fiber at $\zeta$

$$\pi_j^{-1}(\zeta) = S^j(\mathfrak{g}_\zeta^*)^{G_\zeta},$$

the invariant polynomials of degree $j$ on the Lie algebra of the isotropy group at that point. The action of $G$ on $\widetilde{B}^j$ by conjugation covers the action of $G$ on $X$, hence there is an induced bundle, shown below to be flat, on the quotient,

$$B^j \longrightarrow Z,$$

which we call the *Borel bundle of degree* $j$.

The *Borel bundle* is the inductive limit of these finite dimensional Borel bundles of degree at most $k$,

$$B^* = \varinjlim_k \bigoplus_{j\le k} B^j,$$

and is an infinite dimensional, graded flat vector bundle over $Z$.

This bundle is non-trivial in general, as explained in Remark 1 below.

The equivariant differential on $\mathcal{C}^\infty_G(X; S(\mathfrak{g}^*)\otimes\Lambda^*)$ descends to the de Rham differential with coefficients in $B^j$ on each complex $\mathcal{C}^\infty(Z; B^j\otimes\Lambda^*)$. The corresponding cohomology groups $\mathrm{H}^*(Z; B^j)$ can be combined to give the cohomology of $Z$ with coefficients in $B^*$ :

$$\mathrm{H}^q(Z; B^*) = \bigoplus_{j+2k=q} \mathrm{H}^j(Z; B^k).$$

We will show that:

**Theorem 4.2.** *If $X$ is a manifold with a smooth $G$ action such that $X = X^{[K]}$, then the absolute equivariant cohomology is naturally isomorphic to the cohomology of the quotient with coefficients in $B^*$*

$$\mathrm{H}^q_G(X) = \mathrm{H}^q(Z; B^*).$$

A similar description of equivariant K-theory, and the delocalized equivariant cohomology, can be given in terms of a flat bundle of rings over $Z$.

*Definition* 2. Let $X$ be a compact manifold with a smooth $G$-action such that $X = X^{[K]}$. Let $\widetilde{\mathcal{R}} \longrightarrow X$ be the fiber bundle over $X$ with fiber the representation ring of the isotropy group of that point, $\mathcal{R}(G_\zeta)$. The action of $G$ on the conjugacy class of $K$ induces an action of $G$ on $\widetilde{\mathcal{R}}$ which covers the action of $G$ on $X$; the induced flat ring bundle over the quotient, $\mathcal{R} \longrightarrow Z$, is termed the *representation bundle.*

In case $K$ is a normal subgroup of $G$, and with slight modification otherwise, there is a normal subgroup $G'$ of $G$ with $G/G'$ a finite group acting on $K$ and such that

$$\mathcal{R} = \mathcal{R}(K) \times_{G/G'} (G'\backslash X). \tag{4.2}$$

Thus in this case $\mathcal{R}$ can be thought of as a fiber bundle over $Z$ with structure group $G/G'$ and with a ring structure on the fibers.

Standard complex $K$-theory over a manifold can be extended to the case of coefficients in a ring, resulting in the tensor product with the cohomology, and can be extended further to the case of a flat bundle of discrete rings over the space.

*Definition* 3. Let $\mathscr{R} \longrightarrow X$ be a flat bundle of discrete rings. We define a vector bundle with coefficients in $\mathscr{R}$, $V \xrightarrow{p} X$, as follows. Each $\zeta \in X$ has a neighborhood $\mathcal{U}$ over which $\mathscr{R}$ is isomorphic to $\mathscr{R}_\zeta \times \mathcal{U}$ and, over this neighborhood, $V$ is an element of $\mathscr{R}_\zeta \otimes \mathrm{Vect}(\mathcal{U})$ associating to each generator of $\mathscr{R}_\zeta$ a vector bundle over $\mathcal{U}$ with only finitely many of these bundles non-zero. Transition maps are bundle isomorphisms combined with the transition maps of $\mathscr{R}$.

Even K-theory with coefficients in $\mathscr{R}$, $\mathrm{K}^0(X;\mathscr{R})$, is the Grothendieck group based on vector bundles with coefficients in $\mathscr{R}$, and the corresponding odd K-theory groups are defined by suspension as in (3.4).

In particular the groups $\mathrm{K}^*(Z;\mathcal{R})$ arise from the representation bundle over the quotient by a group action with unique isotropy type. In view of (4.2), when $K$ is normal these groups are

$$\mathrm{K}^*(Z;\mathcal{R}) = (\mathcal{R}(K) \otimes \mathrm{K}^*(G' \backslash X))^{G/G'}. \tag{4.3}$$

**Theorem 4.3.** *If $X$ be a compact manifold with a smooth $G$-action such that $X = X^{[K]}$, then there is a natural isomorphism*

$$\mathrm{K}^*_G(X) = \mathrm{K}^*(Z;\mathcal{R}).$$

Given this description of equivariant K-theory it is natural to 'interpolate' between equivariant cohomology and equivariant K-theory and consider cohomology with coefficients in $\mathcal{R}$.

*Definition* 4. For a compact manifold with $G$-action with unique isotropy type, the *delocalized equivariant cohomology* of $X$ is the $\mathbb{Z}_2$ graded cohomology

$$\mathrm{H}^{\mathrm{even}}_{G,\mathrm{dl}}(X) = \mathrm{H}^{\mathrm{even}}(Z;\mathcal{R}),\ \mathrm{H}^{\mathrm{odd}}_{G,\mathrm{dl}}(X) = \mathrm{H}^{\mathrm{odd}}(Z;\mathcal{R}).$$

There is a natural localization map from the representation bundle to the Borel bundle over $Z$. Namely, a representation of $K$ can be identified with its character in $\mathcal{C}^\infty(K)^K$, with value at $g \in K$ the trace of the action of $g$, and this is mapped to the invariant polynomial on the Lie algebra determined by its Taylor series at the identity,

$$L : \mathcal{R}(K) = \mathcal{C}^\infty(K)^K \longrightarrow S(\mathfrak{k}^*)^K. \tag{4.4}$$

This map induces a map between the bundles $\widetilde{\mathcal{R}}$ and $\widetilde{B}^* = \oplus \widetilde{B}^j$ over $X$ which is equivariant with respect to the adjoint $G$-action, hence it induces a map

$$L : \mathcal{R} \longrightarrow B^* \tag{4.5}$$

between the representation and Borel bundles over $Z$. This map in turn induces a map between the cohomology of $Z$ with coefficients in $\mathcal{R}$ and the cohomology of $Z$ with coefficients in $B^*$.

**Theorem 4.4.** *If $X$ be a compact manifold with a smooth $G$-action such that $X = X^{[K]}$ there are natural Chern character maps such that*
(4.6)

$$\begin{tikzcd}
\mathrm{K}^0_G(X)\otimes\mathbb{C} = \mathrm{K}^0(Z;\mathcal{R})\otimes\mathbb{C} \arrow[rr, "\mathrm{Ch}_G"] \arrow[dr, "\mathrm{Ch}_G", "\cong"'] & & \mathrm{H}^{\mathrm{even}}_G(X) = \mathrm{H}^{\mathrm{even}}(Z;B^*) \\
& \mathrm{H}^{\mathrm{even}}_{G,\mathrm{dl}}(X) = \mathrm{H}^{\mathrm{even}}(Z;\mathcal{R}) \arrow[ur, "L"] &
\end{tikzcd}$$

*commutes.*

The rest of the section is devoted to proving these theorems. For the purpose of exposition, we work up to the general case and start by treating two extreme cases.

4.1. **Trivial actions.** If a Lie group acts trivially on a space then the chain space for equivariant cohomology becomes $S(\mathfrak{g}^*)^G\otimes\mathcal{C}^\infty(X;\Lambda^*)$, the differential reduces to the untwisted deRham differential and the polynomial coefficients commute with the differential. It follows that the equivariant cohomology groups are given by the finite, graded, tensor product

$$\mathrm{H}^*_G(X) = S(\mathfrak{g}^*)^G\widehat{\otimes}\,\mathrm{H}^*(X). \tag{4.7}$$

Thus the equivariant cohomology group of degree $q$ is given by

$$\mathrm{H}^q_G(X) = \bigoplus_{2j+k=q} S^j(\mathfrak{g}^*)^G\otimes\mathrm{H}^k(X),$$

in accordance with Theorem 4.2 as the Borel bundle in this case is the trivial bundle $S(\mathfrak{g}^*)^G\times X\longrightarrow X$.

There is a similar description of the equivariant K-theory. If $E\longrightarrow X$ is an equivariant vector bundle then each fiber $E_\zeta$ has an induced $G$-action, and hence a decomposition into a combination of irreducible $G$-representations

$$E_\zeta \cong \sum_{\tau\in\widehat{G}} V_\tau\otimes\mathrm{Hom}_G(V_\tau,E_\zeta) \tag{4.8}$$

by the Peter-Weyl theorem. This decomposition extends to a decomposition of $E$ as an element of $\mathcal{R}(G)\otimes\mathrm{K}^0_G(X)$ and this leads to an isomorphism

$$\mathrm{K}^0_G(X)\cong\mathcal{R}(G)\otimes\mathrm{K}^0(X),$$

as expected from Theorem 4.3. Suspension yields a similar description of $\mathrm{K}^1_G(X)$.

In this case the structure of the Chern character can be seen explicitly. Namely, from naturality considerations, it reduces to the ordinary Chern character at the identity and has appropriate $G$-equivariance, so decomposes as a tensor product

$$\mathrm{Ch}_G = L^\wedge\otimes\mathrm{Ch} : \mathcal{R}(G)\otimes\mathrm{K}^0(X)\longrightarrow S(\mathfrak{g}^*)^G\otimes\mathrm{H}^{\mathrm{even}}(X) \tag{4.9}$$

where the localization map $L\wedge$ is given by (4.4).

This also reveals the problem with the Chern character (3.6). Namely if $G$ is not connected, then there may well be a representation with all tensor powers non-trivial – for instance the standard representation of a copy of $\mathbb{Z}_2$ – but with trace which is constant on the component of the identity. The formal difference between this representation and the trivial one-dimensional representation survives in the tensor product with $\mathbb{C}$ but is annihilated by the localization map. Even for a connected group, this phenomenon may arise from isotropy groups.

The definition of the delocalized equivariant cohomology groups is simply

$$\mathrm{H}^{\mathrm{even}}_{G,\mathrm{dl}}(X) = \mathcal{R}(G) \otimes \mathrm{H}^{\mathrm{even}}(X), \quad \mathrm{H}^{\mathrm{odd}}_{G,\mathrm{dl}}(X) = \mathcal{R}(G) \otimes \mathrm{H}^{\mathrm{odd}}(X),$$

and these groups lead to the diagram (4.6).

4.2. **Free actions.** The opposite extreme of the trivial case, studied extensively by Cartan, arises when $G$ acts freely, i.e., when $g\zeta = \zeta$ implies $g = \mathrm{Id}$. The assumed smoothness and compactness shows that the action corresponds to a principal $G$-bundle,

$$\begin{tikzcd} G \arrow[r, dash] & X \arrow[d] \\ & Z = G\backslash X. \end{tikzcd} \tag{4.10}$$

In this case Borel showed that not only is the quotient of the action smooth but

$$G \text{ acts freely} \Longrightarrow \mathrm{H}^*_G(X) = \mathrm{H}^*(Z). \tag{4.11}$$

This indeed is one justification for the definition of equivariant cohomology.

The isomorphism in (4.11) was analyzed explicitly by Cartan at the chain level, in terms of a connection on $X$ as a principal bundle over $X$ (see [12]). Thus, if $\theta$ is a connection on the principal bundle then its curvature, $\omega$, is a 2-form with values in the tensor product of the Lie algebra with itself. The formally infinite sum $\exp(\omega/2\pi i)$ can therefore be paired with an element of the finite part of the symmetric tensor product $S(\mathfrak{g}^*)$; the $G$-invariant part of the resulting form descends to the quotient and gives a map at the form level

$$\left(\mathcal{C}^\infty(X; \Lambda^* X) \otimes S(\mathfrak{g}^*)\right)^G \ni u \longmapsto \left(\exp(\omega/2\pi i) \cdot u\right)^G \in \mathcal{C}^\infty(G\backslash X; \Lambda^*) \tag{4.12}$$

which induces the isomorphism (4.11).

In this free case the equivariant K-theory is also immediately computable. A $G$-equivariant vector bundle over $X$ is equivariantly isomorphic to the pull-back of a vector bundle over $Z$ and in consequence

$$G \text{ acts freely} \Longrightarrow \mathrm{K}^*_G(X) = \mathrm{K}^*(Z). \tag{4.13}$$

In this setting the equivariant Chern character reduces to the standard Chern character on the quotient and in particular the equivariant form of the Atiyah-Hirzebruch isomorphism does hold.

For free group actions, the delocalized equivariant cohomology is just the cohomology of the quotient

$$\mathrm{H}^{\mathrm{even}}_{G,\mathrm{dl}}(X) = \mathrm{H}^{\mathrm{even}}(Z), \quad \mathrm{H}^{\mathrm{odd}}_{G,\mathrm{dl}}(X) = \mathrm{H}^{\mathrm{odd}}(Z)$$

and again the diagram (4.6) is clear, with $L^\wedge$ given by the identity.

4.3. **Unique isotropy group.** The free case corresponds to $G_\zeta = \{\mathrm{Id}\}$. Perhaps the next most regular case is where $G_\zeta = K$ is a fixed group. In view of (2.3),

$$G_\zeta = K \ \forall \ \zeta \in X \Longrightarrow K \text{ normal in } G. \tag{4.14}$$

The action of $G$ factors through the free action of $Q = G/K$. If $G = K \times Q$, we can treat the two actions separately as above. However, generally there will be a non-trivial induced action of $Q$ on the Lie algebra of $K$.

It is convenient to pass to a finite index subgroup of $G$. Let $\pi : G \longrightarrow Q$ denote the canonical projection and let $G'$ be the inverse image of $Q_0$, the connected

component of the identity in $Q$. Then $G'$ is a normal subgroup of $G$ with quotient $G/G' = Q/Q_0$, a finite group.

The Lie algebra $\mathfrak{k}$ of $K$ is a Lie subalgebra of $\mathfrak{g}$. Choosing an Ad-invariant metric on $\mathfrak{g}$, the Lie algebra, $\mathfrak{q}$, of $Q$ may be identified with $\mathfrak{k}^\perp$. The Ad-invariance of $\mathfrak{k}$ implies the Ad-invariance of $\mathfrak{q}$; thus $\mathfrak{k}$ and $\mathfrak{q}$ Lie commute. Exponentiating the Lie algebra $\mathfrak{q}$ into $G'$ gives a subgroup $Q_0' \subset G'$ which is a finite cover of $Q_0$ and which commutes with $K$.

On the one hand, this shows that the Borel bundle of degree $j$, given initially by

$$B^j = S^j(\mathfrak{k}^*)^K \times_G X \longrightarrow Z$$

is equal to

$$S^j(\mathfrak{k}^*)^K \times_{(G/G')} (Q_0 \backslash X) \longrightarrow Z. \tag{4.15}$$

On the other hand, it follows that the equivariant differential forms on $X$ are given by

$$\begin{aligned}(S(\mathfrak{g}^*) \otimes \mathcal{C}^\infty(X;\Lambda))^G &= (S(\mathfrak{k}^*) \otimes S(\mathfrak{q}^*) \otimes \mathcal{C}^\infty(X;\Lambda))^G \\ &= \left(S(\mathfrak{k}^*)^K \otimes (S(\mathfrak{q}^*) \otimes \mathcal{C}^\infty(X;\Lambda))^{Q_0}\right)^{G/G'}.\end{aligned}$$

The equivariant differential reduces to that of the $Q_0$ action and so, using the result for principal bundles above, the equivariant cohomology of $X$ is given by

$$\mathrm{H}^q_G(X) = \bigoplus_{2j+k=q} \left(S^j(\mathfrak{k}^*)^K \otimes \mathrm{H}^k(Q_0 \backslash X)\right)^{G/G'}.$$

In view of (4.15), this can be written

$$\mathrm{H}^q_G(X) = \bigoplus_{2j+k=q} \mathrm{H}^k(Z; B^j), \text{ i.e. } \mathrm{H}^*_G(X) = \mathrm{H}^*(Z; B^*),$$

thus establishing Theorem 4.2 in this case.

*Remark* 1. The Borel bundle is not trivial in general, precisely because of the possible non-trivial action of $Q/Q_0$ on the Lie algebra of $K$. For instance, let $M$ be the connected double cover of the circle with the associated free $\mathbb{Z}_2$ action. This action can be extended to an action of the twisted product $G = \mathbb{S}^1 \rtimes \mathbb{Z}_2$ (where commuting the non-trivial $\mathbb{Z}_2$ element past an element $z \in \mathbb{S}^1 \subseteq \mathbb{C}$ replaces $z$ with $\overline{z}$) since the twisting does not affect the $\mathbb{Z}_2$ product. Now $M$ has a unique isotropy group, $K = \mathbb{S}^1$, which is normal in $G$. The quotient group $Q = G/K$ is $\mathbb{Z}_2$ and it acts non-trivially on $\mathbb{R}^1$, the Lie algebra of $K$. The Borel bundle will thus be a non-trivial line bundle on $\mathbb{Z}_2 \backslash M = \mathbb{S}^1$.

There is a similar reduction for equivariant K-theory in this case of a single isotropy group. Indeed, since $Q_0'$ commutes with $K$, if $Q$ is connected then there are two independent actions, a free action by $Q$ and a trivial action by $K$, so from the previous two sections,

$$\mathrm{K}^*_G(X) = \mathcal{R}(K) \otimes \mathrm{K}^*(Z).$$

If the quotient is not connected, then $K$ still acts on the fibres of any $G$-equivariant vector bundle over $X$, which can then be decomposed into subbundles tensored with representations of $K$, so an equivariant K-class is represented by a pair of equivariant bundles, or a bundle with coefficients in the representation ring $\mathcal{R}(K)$.

Again the quotient group $Q$ acts, by conjugation, on $\mathcal{R}(K)$ with $Q_0$ acting trivially and we may identify

$$\mathrm{K}^*_G(X) = (\mathcal{R}(K) \otimes \mathrm{K}^*(Q_0\backslash X))^{Q/Q_0} \text{ or } \mathrm{K}^*_G = \mathrm{K}^*(Z;\mathcal{R}) \tag{4.16}$$

where in the second, more geometric, form we use the representation bundle, given in this case by

$$\mathcal{R} = \mathcal{R}(K) \times_{G/G'} (Q_0\backslash X) \longrightarrow Z.$$

Note that this establishes (4.2) and (4.3), hence Theorem 4.3, since $Q_0\backslash X = G'\backslash X$ and $G/G' = Q/Q_0$.

In the same way, the delocalized equivariant cohomology of $X$ is given by

$$\mathrm{H}^{\mathrm{even}}_{G,\mathrm{dl}}(X) = (\mathcal{R}(K) \otimes \mathrm{H}^{\mathrm{even}}(Q_0\backslash X))^{Q/Q_0},$$
$$\mathrm{H}^{\mathrm{odd}}_{G,\mathrm{dl}}(X) = \Big(\mathcal{R}(K) \otimes \mathrm{H}^{\mathrm{odd}}(Q_0\backslash X)\Big)^{Q/Q_0}.$$

The Chern character map $\mathrm{K}^0_G(X) \longrightarrow \mathrm{H}^{\mathrm{even}}_{G,\mathrm{dl}}(X)$ is thus induced by the Chern character map $\mathrm{K}^0(Q_0\backslash X) \longrightarrow \mathrm{H}^{\mathrm{even}}(Q_0\backslash X)$ and is an isomorphism after tensoring $\mathrm{K}^0_G(X)$ with $\mathbb{C}$. This establishes Theorem 4.4 when $X = X^K$.

4.4. **Unique isotropy type.** Finally, consider the general case of a unique isotropy type, $X = X^{[K]}$. Denote $N(K)/K$ by $W(K)$, and recall from Proposition 4.1 that $X$ is $G$-equivariantly diffeomorphic to $G \times_{N(K)} X^K$ and that the $G$-orbit space of $X$ is equal to the $W(K)$-orbit space of $X^K$,

$$Z = G\backslash X = W(K)\backslash X^K.$$

Notice that, if $\widetilde{K}$ is another choice of isotropy group, so that $\widetilde{K} = gKg^{-1}$ for some $g \in G$, then $gN(K)g^{-1} = N(\widetilde{K})$ and the diffeomorphism

$$g\cdot : X^K \longrightarrow X^{\widetilde{K}}$$

intertwines the $N(K)$ and $N(\widetilde{K})$ actions and hence descends to a diffeomorphism

$$W(K)\backslash X^K \longrightarrow W(\widetilde{K})\backslash X^{\widetilde{K}}.$$

Directly from the definitions, this shows that the Borel bundle of the $G$-action on $X$ is the same as the Borel bundle of the $N(K)$-action on $X^K$, and similarly for the representation bundle.

The $G$-equivariant cohomology of $X$ is equal to the $N(K)$-equivariant cohomology of $X^K$ (as observed originally by Borel). Indeed, the space $G \times X^K$ has two commuting free actions, the left $G$-action and the diagonal $N(K)$-action, hence

$$\mathrm{H}^*_G(X) = \mathrm{H}^*_G(G \times_{N(K)} X^K) = \mathrm{H}_{G\times N(K)}(G \times X^K) = H^*_{N(K)}(X^K).$$

By the previous section, this is equal to

$$H^*_{N(K)}(X^K) = \mathrm{H}^*(W(K)\backslash X^K; B^*) = \mathrm{H}^*(Z; B^*),$$

thus establishing Theorem 4.2.

Since $K$-theory behaves similarly for free group actions, the $G$-equivariant K-theory of $X$ also reduces to the $N(K)$-equivariant K-theory of $X^K$,

$$\mathrm{K}^0_G(X) = \mathrm{K}^0_{N(K)}(X^K) = \mathrm{K}^0(Z;\mathcal{R}). \tag{4.17}$$

Finally, since delocalized equivariant cohomology is defined directly on the orbit space, it is immediate that $\mathrm{H}^*_{G,\mathrm{dl}}(X) = \mathrm{H}^*_{N(K),\mathrm{dl}}(X^K)$, which reduces Theorems

4.3 and 4.4 to the situation discussed in the previous section and establishes them when $X = X^{[K]}$.

## 5. Equivariant fibrations

In describing the equivariant cohomology theories for a general group action in terms of the resolved space, we will often have two compact manifolds, $Y_1$ and $Y_2$, each with a smooth $G$-action with unique isotropy type, and with a smooth map

$$f : Y_1 \longrightarrow Y_2 \tag{5.1}$$

which is both $G$-equivariant and a fibration. A $G$-equivariant map always descends to map between the orbit spaces and this induces maps between the Borel and representation bundles.

**Proposition 5.1.** *If $f : Y_1 \longrightarrow Y_2$ is a $G$-equivariant fibration between compact manifolds with $G$-actions having unique isotropy type then the induced map between the quotients $\gamma_f : Z_1 = G\backslash Y_1 \longrightarrow Z_2 = G\backslash Y_2$ is a smooth fibration which is covered by induced pull-back/restriction maps between sections of the Borel and representation bundles*

$$\gamma_f^{\#} : \mathcal{C}^\infty(Z_2, B_2^*) \longrightarrow \mathcal{C}^\infty(Z_1, B_1^*),\ \gamma_f^{\#} : \mathcal{C}^\infty(Z_2, \mathcal{R}_2) \longrightarrow \mathcal{C}^\infty(Z_1, \mathcal{R}_1)$$

*which in turn induce maps between the corresponding chain spaces for equivariant and delocalized equivariant cohomology and for bundles with representation bundle coefficients.*

*Proof.* The smoothness of $\gamma_f : Z_1 \longrightarrow Z_2$, follows from the fact that $G$-invariant smooth functions on the base pull back to smooth $G$-invariant functions, this also shows that $(\gamma_f)^*$ is injective and hence that $\gamma_f$ is a fibration.

For $\zeta \in Z_1$, $G_\zeta \subseteq G_{f(\zeta)}$, and hence there are restriction maps

$$\mathcal{R}(G_{f(\zeta)}) \longrightarrow \mathcal{R}(G_\zeta) \text{ and } S(\mathfrak{g}^*_{f(\zeta)})^{G_{f(\zeta)}} \longrightarrow S(\mathfrak{g}^*_\zeta)^{G_\zeta}. \tag{5.2}$$

Since the Borel bundles of degree $j$ are obtained from the equivariant bundles

$$\widetilde{B}_i^j = \{(\zeta, \omega) \in Z_i \times S(\mathfrak{g}^*_\zeta)^{G_\zeta}\} \longrightarrow Y_i$$

and (5.2) induces an equivariant bundle map

$$f^*(\widetilde{B}_2^k) \longrightarrow \bigoplus_{j \le k} \widetilde{B}_1^j,$$

there are maps

$$\gamma_f^{\#} : \mathcal{C}^\infty(Z_2, B_2^k) \longrightarrow \mathcal{C}^\infty\Big(Z_1, \bigoplus_{j\le k} B_1^j\Big),\ \gamma_f^{\#} : \mathcal{C}^\infty(Z_2, B_2^*) \longrightarrow \mathcal{C}^\infty(Z_1, B_1^*).$$

Similarly, the representation bundles are obtained from the equivariant bundles

$$\widetilde{\mathcal{R}}_i = \{(\zeta, \rho) \in Z_i \times \mathcal{R}(G_\zeta)\} \longrightarrow Y_i$$

and (5.2) induces an equivariant bundle map

$$f^*(\widetilde{\mathcal{R}}_2) \longrightarrow \widetilde{\mathcal{R}}_1,$$

so $\gamma_f^{\#} : \mathcal{C}^\infty(Z_2, \mathcal{R}_2) \longrightarrow \mathcal{C}^\infty(Z_1, \mathcal{R}_1)$, as claimed.

Together with pull-back of forms these generate pull-back maps for the 'reduced' deRham spaces defining the equivariant cohomology and the equivariant delocalized cohomology

$$\begin{aligned} \gamma_f^{\#} &: \mathcal{C}^\infty(Z_2; B_2^* \otimes \Lambda^*) \longrightarrow \mathcal{C}^\infty(Z_1; B_1^* \otimes \Lambda^*) \\ \gamma_f^{\#} &: \mathcal{C}^\infty(Z_2; \mathcal{R}_2 \otimes \Lambda^*) \longrightarrow \mathcal{C}^\infty(Z_1; \mathcal{R}_1 \otimes \Lambda^*) \end{aligned} \tag{5.3}$$

which commute with the differential and so in turn induce pull-back maps on the corresponding cohomologies.

The situation is similar for the reduced model for equivariant K-theory. Namely, since $G_\zeta \subseteq G_{f(\zeta)}$, there is a natural 'Peter-Weyl' pull-back map where each element of $\mathcal{R}_2$ is decomposed into a finite sum of elements of $\mathcal{R}_1$. This induces a pull-back construction, so that if

$$\begin{gathered} V \text{ a bundle with coefficients in } \mathcal{R}_2 \Longrightarrow \\ \gamma_f^{\#} V \text{ is a bundle with coeffiencients in } \mathcal{R}_1. \end{gathered} \tag{5.4}$$

□

These pull-back maps also allow the introduction of relative cohomology and K-theory groups. The situation that arises inductively below corresponds to $Y_1$ being a boundary face of a manifold (with corners) $Y$; for simplicity here suppose that $Y$ is a manifold with boundary. Then the relative theory in cohomology for the pair of quotient spaces, is fixed by the chain spaces

$$\left\{(u,v) \in \mathcal{C}^\infty(G\backslash Y; B^* \otimes \Lambda^*) \times \mathcal{C}^\infty(Z_2; B_2^* \otimes \Lambda^*); i_{Z_1}^* u = \gamma_f^{\#} v\right\} \tag{5.5}$$

with the diagonal differential; here $i_{Z_1}^*$ is the map induced by restriction to the boundary.

Similar considerations apply to delocalized cohomology and K-theory.

## 6. Resolution of a group action and reduction

A connected manifold, $X$, with smooth $G$-action has a single isotropy group which is maximal with respect to the partial order (2.5). Corresponding to this is the open and dense the principal isotropy type, $X_{\mathrm{princ}}$. The space $X$ can be viewed as a compactification of $X_{\mathrm{princ}}$ and in this section we recall from [1] that there is another compactification of $X_{\mathrm{princ}}$ to a manifold with corners (even if $X$ originally had no boundary), $Y(X)$, with a smooth $G$-action, a unique isotropy type, and an equivariant map

$$Y(X) \longrightarrow X,$$

that restricts to a diffeomorphism $Y(X)^\circ \longrightarrow X_{\mathrm{princ}}$. The space $Y(X)$ is canonically associated to $X$ with its $G$-action and is called here the *canonical resolution* of $X$.

In a manifold with corners there is a distinguished class of submanifolds, the 'p-submanifolds' where 'p-' stands for 'product-', which have tubular neighborhoods in the manifold. Radial blow-up of a closed p-submanifold, in the sense of replacing the submanifold with the boundary of a tubular neighborhood, leads to a well defined manifold with corners.

In [1] it is shown that a minimal isotropy type of $X$ with respect to the partial order (2.5) is a closed $G$-invariant p-submanifold, and that blowing it up produces a manifold with a smooth $G$-action in which this minimal isotropy type does not occur. The space $Y(X)$ is obtained by iteratively blowing-up minimal isotropy

types, and is independent of the order in which these blow-ups are carried out. We refer to [1] for the details of the construction and just review the structure of $Y(X)$ that we use below.

*Definition* 5. A manifold with corners $Y$ has a *resolution structure* if every boundary hypersurface $M$ is the total space of a fibration

$$M \xrightarrow{\phi} Y_M$$

and these fibrations are compatible in that:

i) If $M_1$ and $M_2$ are intersecting boundary hypersurfaces then $\dim Y_1 \neq \dim Y_2$.
ii) If $M_1$ and $M_2$ are intersecting boundary hypersurfaces and $\dim Y_1 < \dim Y_2$, there is a given fibration $\phi_{12} : \phi_2(M_1 \cap M_2) \longrightarrow Y_1$ such that the diagram

$$\begin{tikzcd} M_1 \cap M_2 \arrow[rr, "\phi_2"] \arrow[dr, "\phi_1"'] & & \phi_2(M_1 \cap M_2) \arrow[dl, "\phi_{12}"] \\ & Y_1 & \end{tikzcd}$$

commutes.

If all of the spaces involved have $G$-actions and the fibrations are equivariant then this is an *equivariant resolution structure.* If moreover $Y$ and the $Y_M$'s each have a unique isotropy type, we refer to the equivariant resolution structure as a *full resolution.*

The canonical resolution of $X$ is a full resolution in this sense. The conjugacy classes of isotropy groups of $X$, $\{[K_i] \in \mathcal{I}\}$, are in one-to-one correspondence with the boundary hypersurfaces of $Y(X)$. The face corresponding to each $[K]$, $M_{[K]}$, is the total space of a fibration arising from the corresponding blow-down map

$$M_{[K]} \xrightarrow{\phi_{[K]}} Y_{[K]} \tag{6.1}$$

and the base is itself the canonical resolution of the closure of $X^{[K]}$,

$$Y_{[K]} = Y\big(\overline{X^{[K]}}\big).$$

Notice that the (topological) closure of an isotropy type has the 'algebraic' description,

$$\overline{X^{[K]}} = \{\zeta \in X : [K] \leq [G_\zeta]\}.$$

Once the group action has been resolved we can make use of Proposition 4.1. The orbit space of the canonical resolution of $X$,

$$Z(X) = G\backslash Y(X),$$

is thus a smooth manifold with corners, the *reduction* of $X$ and is canonically associated to the $G$-action on $X$. For each isotropy type $[K]$ of $X$, the associated boundary hypersurface of $Y(X)$ (which may not be connected), with its fibration (6.1), descends to a boundary fibration of the corresponding boundary hypersurface of $Z(X)$,

$$\psi_{[K]} : N_{[K]} = G\backslash M_{[K]} \longrightarrow G\backslash Y_{[K]} = Z_{[K]} \tag{6.2}$$

and since the compatibility conditions necessarily descend as well, these fibrations form a resolution structure on $Z(X)$.

## 7. Reduced models for cohomology

Let $X$ be a compact manifold with a smooth action by a compact Lie group, $G$, and let $Y(X)$ and $Z(X)$ be its resolution and reduction. As $Y(X)$ has a unique isotropy type, there are Borel and representation bundles

$$B^* \longrightarrow Z(X), \quad \mathcal{R} \longrightarrow Z(X)$$

as in Definitions 1 and 2. Similarly, for each isotropy type $[K] \in \mathcal{I}$ of $X$, the manifold $Y_{[K]} = Y(\overline{X^{[K]}})$ has a unique isotropy type and there are corresponding Borel and representation bundles

$$B^*_{[K]} \longrightarrow Z_{[K]}, \quad \mathcal{R}_{[K]} \longrightarrow Z_{[K]}.$$

Moreover, as explained in §5, there are natural pull-back/restriction maps under the fibrations (6.2),

$$\psi^{\#}_{[K]} : \mathcal{C}^\infty(Z_{[K]}, B^*_{[K]}) \longrightarrow \mathcal{C}^\infty(N_{[K]}, B^*), \ \ \psi^{\#}_{[K]} : \mathcal{C}^\infty(Z_{[K]}, \mathcal{R}_{[K]}) \longrightarrow \mathcal{C}^\infty(N_{[K]}, \mathcal{R}).$$

This data allows twisted, relative deRham complexes on $Z(X)$ to be introduced.

*Definition* 6. For all $[K] \in \mathcal{I}$, let $i_{[K]} : N_{[K]} \longrightarrow Z(X)$ denote the inclusion of the boundary face of $Z(X)$ corresponding to $[K]$, then for each $q \in \mathbb{N}_0$ set

$$\begin{multline}\tag{7.1} \mathcal{C}^q_\Phi(Z(X); B^* \otimes \Lambda^*) \\ = \Big\{ (u, \{v_{[K]}\}) \in \Big( \bigoplus_{2j+k=q} \mathcal{C}^\infty(Z(X); B^j \otimes \Lambda^k) \Big) \oplus \Big( \bigoplus_{[K]\in\mathcal{I}} \mathcal{C}^\infty(Z_{[K]}; B^*_{[K]} \otimes \Lambda^*) \Big) \\ : i^*_{[K]} u = \psi^{\#}_{[K]} v_{[K]} \text{ for all } [K] \in \mathcal{I} \Big\}. \end{multline}$$

The diagonal deRham differential, $d(u, \{v_{[K]}\}) = (du, \{dv_{[K]}\})$, acts on the sequence $\mathcal{C}^*_\Phi(Z(X); B^* \otimes \Lambda^*)$, increases the grading by one, and squares to zero. Thus there is an associated $\mathbb{Z}$-graded cohomology, the *reduced equivariant cohomology.*

**Theorem 7.1.** *The equivariant cohomology of a compact manifold $X$ (meaning the absolute cohomology in case of a manifold with corners) with a smooth action by a compact Lie group $G$ is equal to the cohomology of the complex $\mathcal{C}^q_\Phi(Z(X); B^* \otimes \Lambda^*)$,*

$$\mathrm{H}^q_G(X) \cong \mathrm{H}^q(\mathcal{C}^*_\Phi(Z(X); B^* \otimes \Lambda^*), d).$$

This result is proved in §7.1; note that in the case of a manifold with corners the additional condition that the action be intersection-free on boundary hypersurfaces is imposed.

We also define reduced K-theory groups on $Z(X)$.

*Definition* 7. A *resolution vector bundle with coefficients in* $\mathcal{R}$ over $Z(X)$ is a collection $\langle E\rangle$ of vector bundles, $E$ over $Z(X)$ with coefficients in $\mathcal{R}$, and one for each non-maximal isotropy type $[K] \in \mathcal{I}$, $E_{[K]}$ over $Z_{[K]}$, with coefficients in $\mathcal{R}_{[K]}$ with the property

$$E\big|_{N_{[K]}} = \phi^{\#}_{[K]} E_{[K]}.$$

The reduced K-theory group $\mathrm{K}^0_\Phi(Z(X), \mathcal{R})$ is the Grothendieck group of pairs of resolution vector bundles with coefficients in $\mathcal{R}$. Equivalence is stable isomorphism over each space consistent under these pull-back maps. The group $\mathrm{K}^1_\Phi(Z(X), \mathcal{R})$ is defined by suspension as in (3.4).

**Theorem 7.2.** *Let $X$ be a closed manifold with a smooth $G$-action, then*

$$\mathrm{K}^*_G(X) \cong \mathrm{K}^*_\Phi(Z(X);\mathcal{R}). \tag{7.2}$$

This result is proved in §7.2.

We defer the definition of the delocalized equivariant cohomology groups $\mathrm{H}^*_{G,\mathrm{dl}}(X)$ for a general $G$-action until section 8 where it is shown that the Chern character map between $\mathrm{K}_G(X)$ and $\mathrm{H}^*_G(X)$ factors through $\mathrm{H}^*_{G,\mathrm{dl}}(X)$.

7.1. **Reduced equivariant cohomology.** To prove Theorem 7.1, we first show that equivariant cohomology lifts to $Y(X)$ and then that it descends to $Z(X)$.

For the former, let $Y$ be any manifold with corners with an equivariant resolution structure, let $\mathcal{M}_1(Y)$ denote the set of boundary hypersurfaces of $Y$, and define

$$\begin{aligned}\mathcal{C}^*_{G,\Phi}(Y;\Lambda^*)&\\ &= \Big\{(u,\{v_M\}) \in \mathcal{C}^\infty_G(Y;S(\mathfrak{g}^*)\otimes\Lambda^*) \oplus \bigoplus_{M\in\mathcal{M}_1(Y)} \mathcal{C}^\infty_G(Y_M;S(\mathfrak{g}^*)\otimes\Lambda^*)\\ &\qquad\qquad : i^*_M u = \phi^*_M v_M \text{ for all } M \in \mathcal{M}_1(Y)\Big\}.\end{aligned}$$

These spaces form a complex with respect to the diagonal equivariant differential, the resulting cohomology is the *equivariant resolution cohomology of* $Y$ and is denoted $\mathrm{H}^*_{G,\Phi}(Y)$.

Let $W$ be a $G$-invariant p-submanifold of $Y$ transversal to the fibers of all of the boundary fibrations of $Y$ then the manifold $[Y;W]$, obtained from $Y$ by blowing-up $W$, is a manifold with corners with an induced equivariant resolution structure. Indeed, the new boundary face, which we denote $\mathrm{ff}[Y;W]$ or ff, can be identified with the inward-pointing spherical normal bundle of $W$ in $Y$ and so is the total space of a fibration over $W$. If $M$ is a boundary hypersurface of $Y$ then $[M;M\cap W]$ is the associated boundary hypersurface of $[Y;W]$ and the fibration on $M$ induces a fibration on $[M;M\cap W]$. Finally the compatibility conditions on the boundary fibrations of $Y$ induce the compatibility conditions on the boundary fibrations of $[Y;W]$. We refer to [1] for the details.

**Proposition 7.3.** *Let $Y$ be a manifold with corners with an equivariant resolution structure and let $W$ be a $G$-invariant p-submanifold transverse to the fibers of all of the boundary fibrations of $Y$, then the equivariant resolution cohomology groups of $Y$ and of $[Y;W]$ are canonically isomorphic,*

$$\mathrm{H}^*_{G,\Phi}(Y) \cong \mathrm{H}^*_{G,\Phi}([Y;W]).$$

*Proof.* Let $\beta : [Y;W] \longrightarrow Y$ denote the blow-down map, which collapses the new boundary face back to $W$. The boundary fibration on $\mathrm{ff}[Y;W]$ is the restriction of $\beta$, $\mathrm{ff}[Y:W] \xrightarrow{\beta} W$, and the boundary fibration $[M;M\cap W] \longrightarrow Y_M$ is the composition $[M;M\cap W] \xrightarrow{\beta} M \xrightarrow{\phi_M} Y_M$. So the pull-back by $\beta$ induces a map

$$\mathcal{C}^*_{G,\Phi}(Y;\Lambda^*) \ni (u,\{v_M\}) \mapsto (\beta^*u, v_{\mathrm{ff}}, \{v_M\}) \in \mathcal{C}^*_{G,\Phi}([Y;W];\Lambda^*)$$

where $v_{\mathrm{ff}}$ is the restriction of $u$ to $\mathrm{ff}[Y;W]$. This map commutes with the diagonal equivariant differential.

The principal claim is that the complex $\mathcal{C}^*_{G,\Phi}([Y;W];\Lambda^*)$ retracts onto the lift of the complex $\mathcal{C}^*_{G,\Phi}(Y;\Lambda^*)$, so they have the same cohomology.

Indeed, the blow-down map has the property that its restriction off the front face is a diffeomorphism,

$$\beta' : [Y;W] \setminus \mathrm{ff}[Y;W] \longrightarrow Y \setminus W$$

Hence given $(u, v_{\mathrm{ff}}, \{v_M\}) \in \mathcal{C}^*_{G,\Phi}([Y;W];\Lambda^*)$, $u$ can be restricted to $[Y;W] \setminus \mathrm{ff}[Y;W]$ and then pulled back along $(\beta')^{-1}$ to give a form $u'$ on $Y \setminus W$. The fact that $i^*_{\mathrm{ff}} u = \beta^* v_{\mathrm{ff}}$ shows that $u'$ extends continuously to $Y$ with $i^*_W u' = v_{\mathrm{ff}}$; however this form is generally not smooth at $W$, and hence does not define an element of $\mathcal{C}^*_{G,\Phi}(Y;\Lambda^*)$. The remedy is to choose a collar neighborhood of $\mathrm{ff}[Y;W]$ in $[Y;W]$ of the form $[0,1)_r \times \mathrm{ff}[Y;W]$ and first deform $u$ to make it independent of $r$ in a neighborhood of $\mathrm{ff}[Y;W]$. The form $u'$ will then extend smoothly to $Y$, and hence define an element of $\mathcal{C}^*_{G,\Phi}(Y;\Lambda^*)$. This proves that the complex on $[Y;W]$ retracts onto the lift of the complex on $Y$, as required. □

This proposition can be applied repeatedly through the resolution of a manifold $X$ to $Y(X)$ starting with the Cartan model for the equivariant cohomology of $X$. At each step of the construction, it is shown in [1] that the conditions of Proposition 7.3 are satisfied, that one has a manifold with corners with an equivariant resolution structure, and that any minimal isotropy type is a closed $G$-invariant p-submanifold transverse to the fibers of all of the boundary fibrations.

**Corollary 7.4.** *The equivariant cohomology of a compact manifold $X$ with a smooth $G$-structure is naturally isomorphic to the equivariant resolution cohomology of its canonical resolution,*

$$\mathrm{H}^q_G(X) \cong \mathrm{H}^q_{G,\Phi}(Y(X)).$$

The projections $\pi : Y(X) \longrightarrow G\backslash Y(X) = Z(X)$ and $\pi_{[K]} : Y_{[K]} \longrightarrow Z_{[K]}$ induce a pull-back map

$$\pi^{\#} : \mathcal{C}^*_{\Phi}(Z(X); B^* \otimes \Lambda^*) \longrightarrow \mathcal{C}^*_{G,\Phi}(Y(X);\Lambda^*)$$

and we can generalize (4.11) to this context. As it is not important for this purpose that $Y(X)$ comes from resolving $X$, we will not assume that in the following.

**Proposition 7.5.** *Let $Y$ be a manifold with corners with a smooth $G$ action for which the orbit of any boundary hypersurface is embedded. If $Y$ is a full resolution in the sense of Definition 5 and $Z = G\backslash Y$ is its orbit space with the induced resolution structure, then pull-back via the projection $\pi : Y \longrightarrow Z$ induces an isomorphism in cohomology,*

$$\pi^{\#} : \mathrm{H}^*_{\Phi}(Z; B^*) \longrightarrow \mathrm{H}^*_{G,\Phi}(Y).$$

*Proof.* To prove that $\pi^{\#}$ induces an isomorphism in cohomology we pass to a relative form of both the complex defining the equivariant resolution cohomology of $Y$ and the complex defining the resolution cohomology of $Z$. Thus let $\mathcal{B}_Y \subseteq \mathcal{M}_1(Y)$ be a $G$-invariant collection of boundary hypersurfaces with the property that it contains any boundary face which corresponds to an isotropy type containing the isotropy group of an element of $\mathcal{B}_Y$. Let $\mathcal{B}_Z$ denote the corresponding collection of boundary hypersurfaces of $Z$. Then consider the subcomplex

$$\mathcal{C}^*_{G,\Phi}(Y;\Lambda^*;\mathcal{B}_Y) = \{(u,\{v_M\}) \in \mathcal{C}^*_{G,\Phi}(Y;\Lambda^*) : M \in \mathcal{B}_Y \implies v_M = 0\}$$

and similarly $\mathcal{C}^*_{\Phi}(Z; B^*\otimes\Lambda^*;\mathcal{B}_Z)$. Again the pull-back map $\pi^{\#}$ acts from the reduced complex to the resolved complex.

In the case that $\mathcal{B}_Y = \mathcal{M}_1(Y)$ we already know from Theorem 4.2 that $\pi^\#$ induces an isomorphism in cohomology since the proof extends with no essential changes to the relative equivariant cohomology of a manifold with corners with a unique isotropy type.

Now, consider two such subsets $\mathcal{B}_Z \subset \mathcal{B}'_Z$ which differ by just one element $N \in \mathcal{M}_1(Z)$, and let $\mathcal{M} = \pi^{-1}(N) \subseteq \mathcal{B}_Y$ and $\mathcal{B}'_Y = \mathcal{B}_Y \setminus \mathcal{M}$. We get two short exact sequence of complexes with maps induced by $\pi^\#$ :

$$\begin{tikzcd} \mathcal{C}_\Phi(Z; B^* \otimes \Lambda^*; \mathcal{B}'_Z) \arrow[r] \arrow[d, "\pi^\#"] & \mathcal{C}_\Phi(Z; B^* \otimes \Lambda^*; \mathcal{B}_Z) \arrow[r] \arrow[d, "\pi^\#"] & \mathcal{C}_\Phi(N; B^* \otimes \Lambda^*; \mathcal{B}_Z) \arrow[d, "\pi^\#"] \\ \mathcal{C}^\infty_{G,\Phi}(Y; \Lambda^*; \mathcal{B}'_Y) \arrow[r] & \mathcal{C}^\infty_{G,\Phi}(Y; \Lambda^*; \mathcal{B}_Y) \arrow[r] & \mathcal{C}^\infty_{G,\Phi}(\mathcal{M}; \Lambda^*; \mathcal{B}_Y). \end{tikzcd} \tag{7.3}$$

Here of course $\mathcal{M}$ is really the orbit of one hypersurface $M \in \mathcal{M}_1(Y)$ under the $G$-action. Because of the assumptions on the action, this orbit is a disjoint collection of boundary hypersurfaces of $Y$.

Now, proceeding by induction over the dimension of $Y$ we may assume that $\pi^\#$ induces an isomorphism on cohomology when acting on $N$. Also inductively, starting from $\mathcal{M}_1(Y)$, we may assume that it induces an isomorphism for the cohomology relative to $\mathcal{B}'_Y$. Thus the Fives Lemma applies to the long exact sequence in cohomology to show that it also induces an isomorphism on cohomology relative to $\mathcal{B}$ and hence in general. □

Corollary 7.4 and Proposition 7.5 together imply Theorem 7.1.

7.2. **Reduced equivariant K-theory.** The equivariant K-theory of a closed manifold $X$ with a smooth $G$-action also lifts to $Y(X)$ and descends to $Z(X)$.

*Definition* 8. Let $Y$ be a manifold with corners and an equivariant resolution structure. An *equivariant resolution vector bundle* over $Y$ is a collection $\langle E\rangle$ of equivariant vector bundles, $E$ over $Y$, and an $E_M$ over each $M \in \mathcal{M}_1(Y)$, with the property that

$$E\big|_M = \phi_M^* E_M \tag{7.4}$$

as equivariant bundles.

The equivariant resolution K-theory group $\mathrm{K}^0_{G,\Phi}(Y)$ is the Grothendieck group of pairs of equivariant resolution vector bundles over $Y$. Equivalence is equivariant stable isomorphism over each space consistent under these pull-back maps. The group $\mathrm{K}^1_{G,\Phi}(Y)$ is defined by suspension as in (3.4).

**Proposition 7.6.** *Let $Y$ be a manifold with corners with an equivariant resolution structure and let $W$ be a $G$-invariant p-submanifold transverse to the fibers of all of the boundary fibrations of $Y$, then the equivariant resolution K-theory groups of $Y$ and of $[Y; W]$ are canonically isomorphic,*

$$\mathrm{K}^*_{G,\Phi}(Y) \cong \mathrm{K}^*_{G,\Phi}([Y; W]).$$

*Proof.* For continuous data this is immediate since the lift of a $G$-equivariant bundle from $Y$ to $[Y; W]$ gives a bundle satisfying the compatibility conditions (7.4). Conversely such data defines a *continuous* bundle over $Y$ by collapsing $W$; thus there is an isomorphism at the level of the bundle data.

For smooth data this is not quite the case, a smooth bundle certainly lifts to give smooth compatible data on the resolution but the converse does not hold. Nevertheless, normal retraction easily shows that any smooth compatible data on the resolution can be deformed by $G$-equivariant homotopy, and hence $G$-equivariant isomorphism, to be the lift of a smooth $G$-bundle over $Y$. The same argument applies to equivalence so the smooth equivariant resolution K-theory groups for $Y$ and $[Y;W]$ are again canonically isomorphic. □

**Corollary 7.7.** *The equivariant K-theory groups of a compact manifold $X$ with a smooth $G$-action are canonically isomorphic to the equivariant resolution K-theory groups of its resolution,*

$$\mathrm{K}^*_G(X) \cong \mathrm{K}^*_{G,\Phi}(Y(X)). \tag{7.5}$$

As for equivariant cohomology, the projection map $\pi : Y(X) \longrightarrow Z(X)$ induces a pull-back map

$$\pi^\# : \mathrm{K}^*_\Phi(Z(X);\mathcal{R}) \longrightarrow \mathrm{K}^*_{G,\Phi}(Y(X))$$

which is in fact an isomorphism.

**Proposition 7.8.** *Let $Y$ be a manifold with corners with a smooth $G$ action. If $Y$ is a full resolution in the sense of Definition 5 and $Z = G\backslash Y$ is its orbit space with the induced resolution structure, then pull-back via the projection $\pi : Y \longrightarrow Z$ induces an isomorphism in cohomology,*

$$\pi^\# : \mathrm{K}^*_\Phi(Z;\mathcal{R}) \longrightarrow \mathrm{K}^*_{G,\Phi}(Y).$$

*Proof.* The proof is essentially the same as that of Proposition 7.5 and we adopt the same notations, e.g., $\mathcal{B}_Y$ and $\mathcal{B}_Z$.

The relative group $\mathrm{K}_{G,\Phi}(Y;\mathcal{B}_Y)$ is made up of equivalence classes $(\langle E\rangle, \langle F\rangle, \sigma)$, where $\langle E\rangle$ and $\langle F\rangle$ are equivariant resolution vector bundles and $\sigma$ is a collection of equivariant isomorphisms

$$\sigma_M : E_M \xrightarrow{\cong} F_M, \text{ for every } M \in \mathcal{B}_Y.$$

Similarly, the relative group $\mathrm{K}_\Phi(Z;\mathcal{R};\mathcal{B}_Z)$ consists of equivalence classes of triples $(\langle E\rangle, \langle F\rangle, \sigma)$, where $\langle E\rangle$ and $\langle F\rangle$ are resolution vector bundles with coefficients in $\mathcal{R}$, and $\sigma$ is a collection of isomorphisms

$$\sigma_N : E_N \xrightarrow{\cong} F_N \text{ as bundles with } \mathcal{R}_N \text{ coefficients for every } N \in \mathcal{B}_Z.$$

The relative odd K-theory groups are defined similarly.

For $\mathcal{B}_Y = \mathcal{M}_1(Y)$, a simple modification of the proof of Theorem 4.3 shows that

$$\pi^\# : \mathrm{K}^*_\Phi(Z;\mathcal{R};\mathcal{B}_Z) \longrightarrow \mathrm{K}^*_{G,\Phi}(Y;\mathcal{B}_Y)$$

is an isomorphism. For the general case, one can appeal to the Fives Lemma in the associated long exact sequences and induction, just as in the proof of Proposition 7.5. □

## 8. Equivariant Chern character

Comparing the reduced equivariant cohomology groups and the reduced equivariant K-theory groups, it is natural to define a complex on $Z(X)$ making use of

the representation bundles,

$$
\begin{aligned}
(8.1)\quad & \mathcal{C}^{\mathrm{even}}_{\Phi}(Z(X);\mathcal{R}\otimes\Lambda^*) \\
&= \Big\{(u,\{v_{[K]}\}) \in \mathcal{C}^\infty(Z(X);\mathcal{R}\otimes\Lambda^{\mathrm{even}}) \oplus \bigoplus_{[K]\in\mathcal{I}} \mathcal{C}^\infty(Z_{[K]};\mathcal{R}_{[K]}\otimes\Lambda^{\mathrm{even}}) \\
&\qquad\qquad : i^*_{[K]}u = \psi^{\#}_{[K]}v_{[K]} \text{ for all } [K]\in\mathcal{I}\Big\},
\end{aligned}
$$

and similarly $\mathcal{C}^{\mathrm{odd}}_{\Phi}(Z(X);\mathcal{R}\otimes\Lambda^*)$. The differential is again the diagonal deRham differential.

*Definition* 9. The *delocalized equivariant cohomology* of a closed manifold $X$ with a smooth $G$-action is the $\mathbb{Z}_2$-graded cohomology of the complex of $\mathcal{R}$-valued forms on $Z(X)$ compatible with the resolution structure,

$$
\mathrm{H}^*_{G,\mathrm{dl}}(X) = \mathrm{H}^*(\mathcal{C}^*_{\Phi}(Z(X);\mathcal{R}\otimes\Lambda^*), d).
$$

It is not immediately apparent that $\mathrm{H}^*_{G,\mathrm{dl}}(X)$ fixes a contravariant functor for smooth $G$-actions, since in general a smooth $G$-equivariant map between manifolds does not lift to a smooth map between the resolutions of the quotients as defined above. Nevertheless this follows immediately from the following theorem since we can identify these rings with $G$-equivariant K-theory with complex coefficients.

**Theorem 8.1.** *The Chern character, defined locally by a choice of compatible connections, defines a map*

$$
\mathrm{Ch}_G : \mathrm{K}^*_{\Phi}(Z(X);\mathcal{R}) \longrightarrow \mathrm{H}^*_{\Phi}(Z(X);\mathcal{R}) \tag{8.2}
$$

*for any smooth action of a compact Lie group on a manifold and this map induces a (Baum-Brylinski-MacPherson) isomorphism*

$$
\mathrm{Ch}_G : \mathrm{K}^*_{\Phi}(Z(X);\mathcal{R})\otimes\mathbb{C} \longrightarrow \mathrm{H}^*_{\Phi}(Z(X);\mathcal{R}). \tag{8.3}
$$

*Proof.* A compatible connection on an resolution vector bundle with coefficients in $\mathcal{R}$ over $Z(X)$ can be introduced by starting from the 'bottom' of the resolution structure and successively extending. Since the coefficient bundles are flat rings, or by lifting to the finite cover by $G/G'$ at each level, the Chern character is then given by the standard formula

$$
v_{[K]} = \exp(\nabla^2/2\pi i) \in \mathcal{C}^\infty(Z_{[K]};\mathcal{R}\otimes\Lambda^{\mathrm{even}}). \tag{8.4}
$$

These forms are clearly compatible so define the class $\mathrm{Ch}_G \in \mathrm{H}^{\mathrm{even}}_{\Phi}(Z(X);\mathcal{R})$. The standard arguments in Chern-Weil theory show that the resulting class is independent of choice of connection. Thus the even Chern character (8.2) is defined as in the setting of smooth manifolds, and similarly the odd Chern character.

In the case of a manifold with unique isotropy type, (4.3) allows this map to be derived from the standard, untwisted, Chern character. Namely the quotient is then a single manifold with corners and the Chern character as defined above is simply the quotient under the finite group action by $G/G'$ of the standard Chern character

$$
\mathrm{K}^*(N(K)'\backslash X^K) \longrightarrow \mathrm{H}^*(N(K)'\backslash X^K). \tag{8.5}
$$

It therefore follows that it induces an isomorphism as in (8.3) in that case. Moreover, this is equally true for absolute and relative K-theory and cohomology.

The proof that (8.3) holds in general follows the same pattern as the proofs above of the identity of $G$-equivariant K-theory and cohomology with the reduced models. Namely, for K-theory and cohomology the partially relative rings can be defined with respect to any collection of boundary $\mathcal{B} \subset \mathcal{M}_1(G\backslash X)$ which contains all hypersurfaces smaller than any element. In the corresponding long exact sequences in K-theory and delocalized cohomology, which in the second case either can be deduced by analogy from the case of coefficient rings or else itself can be proved inductively, the Chern character then induces a natural transformation by the Fives Lemma. $\square$

Note that the localization maps (4.5) are consistent with the pull-back/restriction maps defined in §5, hence there is a natural localization map

$$L^\wedge : \mathrm{H}^*_{G,\mathrm{dl}}(X) \longrightarrow \mathrm{H}^*_G(X)$$

induced by localizing the complex (8.1) to the complex (7.1), which relates the equivariant Chern character maps on $\mathrm{H}^*_{G,\mathrm{dl}}(X)$ and $\mathrm{H}^*_G(X)$,

$$\begin{tikzcd}[column sep=small] \mathrm{K}^0_G(X)\otimes\mathbb{C} = \mathrm{K}^0_\Phi(Z(X);\mathcal{R})\otimes\mathbb{C} \arrow[rr, "\mathrm{Ch}_G"] \arrow[dr, "\mathrm{Ch}_G"', "\cong"] & & \mathrm{H}^{\mathrm{even}}_G(X) = \mathrm{H}^{\mathrm{even}}_\Phi(Z;B^*) \\ & \mathrm{H}^{\mathrm{even}}_{G,\mathrm{dl}}(X) = \mathrm{H}^{\mathrm{even}}_\Phi(Z;\mathcal{R}) \arrow[ur, "L"'] & \end{tikzcd} \tag{8.6}$$

## 9. Delocalized equivariant cohomology

The determining feature of delocalized equivariant cohomology as compared to Borel's equivariant cohomology is that the Chern character gives an isomorphism as in Theorem 8.1. In this section some of the properties that delocalized equivariant cohomology shares with the more familiar equivariant cohomology are pointed out.

9.1. **Sheaf theoretic description.** As mentioned in the introduction, delocalized equivariant cohomology was introduced by Baum, Brylinski, and MacPherson in the context of Abelian group actions on closed manifolds. Their approach is to define a certain sheaf over $G\backslash X$. The canonical resolution in effect resolves this sheaf to a bundle. Indeed, there is not only a continuous quotient map $p_G : Z(X) \longrightarrow G\backslash X$ but also by iteration continuous maps from all the resolutions of the isotropy types, $p_G : Z_{[K]} \longrightarrow G\backslash X$ which commute with the boundary fibrations. Thus an open subset of $G\backslash X$ lifts to a system of open subsets of $Z(X)$ and all the $Z_{[K]}(X)$ which are iteratively related by the boundary fibrations. It follows that the Borel bundle defines a $\mathbb{Z}$-graded sheaf $\mathrm{B}^*$ over $G\backslash X$, where sections of $\mathrm{B}^*$ over an open set $\mathcal{U} \in \mathrm{Op}(G\backslash X)$ are precisely sections of the Borel bundle of the preimage. With the appropriate grading, the Čech cohomology with coefficients in this sheaf can be identified, by standard arguments, with the cohomology of the reduced Cartan complex, $\mathrm{H}^*_\Phi(Z(X);B^*)$ and hence with $\mathrm{H}^*_G(X)$.

Proceeding in the same way with the representation bundle $\mathcal{R} \longrightarrow Z(X)$, one obtains a sheaf R over $G\backslash X$, with its Čech cohomology identified with the resolution cohomology of $Z(X)$ with coefficients in $\mathcal{R}$, $\mathrm{H}^*_\Phi(Z(X);\mathcal{R})$ and hence with $\mathrm{H}^*_{G,\mathrm{dl}}(X)$.

9.2. **Atiyah-Bott, Berline-Vergne localization.** Among the most useful properties of equivariant cohomology are the localization formulæ of Berline-Vergne [8] and Atiyah-Bott [4] for Abelian group actions. Resolution of a group action reduces these formulæ essentially to Stokes' theorem, and works for non-Abelian actions as well. The extension to non-Abelian group actions was already pointed out by Pedroza-Tu [14] where, following [4], it is deduced from an 'abstract' localization result of Segal as follows.

**Theorem 9.1** (Segal [16])**.** *Let $\gamma$ be a conjugacy class in $G$, $X^\gamma = \cup_{g\in\gamma} X^g$, $i_\gamma : X^\gamma \hookrightarrow X$ the inclusion, and $\langle\gamma\rangle$ the ideal of $R(G)$ consisting of characters that vanish at $\gamma$. Then the pull-back map $i_\gamma^* : \mathrm{K}_G^*(X) \longrightarrow \mathrm{K}_G^*(X^\gamma)$ becomes an isomorphism after localization at $\langle\gamma\rangle$,*

$$i_\gamma^* : \mathrm{K}_G^*(X)_{\langle\gamma\rangle} \xrightarrow{\cong} \mathrm{K}_G^*(X^\gamma)_{\langle\gamma\rangle}. \tag{9.1}$$

This theorem easily implies the analogous statements for equivariant cohomology and delocalized equivariant cohomology. It also implies [4, pg. 8] that the push-forward map $(i_\gamma)_*$ in either cohomology is an isomorphism after localization. The action of $i_\gamma^*(i_\gamma)_*$ is multiplication by the corresponding equivariant Euler class, so this class is a unit after localization. The localization formulæ for integrals follows by considering push-forward along the projections $\pi : X \longrightarrow \mathrm{pt}$, $\pi^\gamma : X^\gamma \longrightarrow \mathrm{pt}$.

**Corollary 9.2** (Berline-Vergne, Atiyah-Bott, Pedroza-Tu)**.** *([8, 4, 5, 14]) Let $\mathrm{h}_G$ denote either equivariant cohomology or delocalized equivariant cohomology and $\chi_G^{\mathrm{h}}(\nu)$ the corresponding Euler class of the normal bundle of $X^\gamma$ in $X$. If $X$ and the normal bundle of $X^\gamma$ are* h*-oriented then, for any $\omega \in \mathrm{h}_G^*(X)$, the equality*

$$\pi_*(\omega) = \pi_*^\gamma\left(\frac{i_\gamma^*\omega}{\chi_G^{\mathrm{h}}(\nu)}\right)$$

*holds after localization.*

Finally we explain how to obtain the integral formula for $\mathrm{H}_G^*(X)$ by means of resolution and Stokes' theorem in the simplest case, since the general case is covered by Corollary 9.2. Assume that $\dim G > 0$ and that the $G$-action on $X$ is free on $X \setminus X^G$. The canonical resolution of $X$, $Y(X)$, is a manifold with boundary and

$$\mathrm{H}_G^*(X) = \mathrm{H}^*(\mathcal{C}_{\Phi,G}^*(Y(X)), d_{eq}).$$

Denote the boundary fibration of $Y(X)$ by $\phi : \partial Y(X) \longrightarrow Y_G$ and the projection $Y(X) \longrightarrow Z(X)$ by $\pi$.

Let $(u, v) \in \mathcal{C}_{\Phi,G}^*(Y(X))$ be $d_{eq}$-closed. By Cartan's isomorphism,

$$u = \pi^* s + d_{eq}\omega$$

(in this case we can assume $s\big|_{\partial Z(X)} = 0$) and by Stokes' theorem

$$\int_{Y(X)} u = \int_{\partial Y(X)} \omega = \int_{Y_G} \int_{\partial Y(X)/Y_G} \omega \quad \text{in } \mathrm{H}_G^*(\mathrm{pt}) = S(\mathfrak{g}^*)^G. \tag{9.2}$$

Next note that

$$d_{eq}\omega\big|_{\partial Y(X)} = (u - \pi^* s)\big|_{\partial Y(X)} = \phi^* v.$$

Hence $(\omega\big|_{\partial Y(X)}, v)$ defines an element of the complex

$$\mathcal{C}^q_G(\partial Y(X),\phi) = \{(\omega,\pi)\in \mathcal{C}^\infty_G(Y_G,\Lambda^*)_{[q+1]}\oplus \mathcal{C}^\infty_G(\partial Y(X),\Lambda^*)_{[q]}\},$$
$$D = \begin{pmatrix} d_{\mathrm{eq}} & 0 \\ \phi^* & -d_{\mathrm{eq}} \end{pmatrix},$$

that computes the equivariant relative cohomology of the normal bundle of $X^G$ in $X$, $N_X X^G$, see [2]. Because $\chi_G(N_X X^G)$ is invertible we can compute the inner integral in (9.2),

$$\int_{\partial Y(X)/Y_G} \omega = v \wedge \chi_G(N_X X^G)^{-1}$$

and hence

$$\int_{Y(X)} u = \int_{Y_G} v\wedge \chi_G(N_X X^G)^{-1}$$

as elements of $H^*_G(\mathrm{pt})$ localized to include $\chi_G(N_X X^G)^{-1}$.

9.3. **Chang-Skjelbred Theorem.** A compact manifold $X$ with a torus $T$ action is equivariantly formal if its equivariant cohomology satisfies

$$\mathrm{H}^*_T(X) = \mathrm{H}(X)\otimes S(\mathfrak{t}^*).$$

**Theorem 9.3** (Knutson-Rosu [15])**.** *Let $X$ be an equivariantly formal $T$-manifold and $X_1$ the set of points in $X$ with stabilizer of codimension at most one, and let $i : X^T \longrightarrow X$ and $j : X^T \longrightarrow X_1$ be the inclusion maps. The induced map*

$$i^* : \mathrm{K}^*_T(X) \longrightarrow \mathrm{K}^*_T(X^T)$$

*is injective and has the same image as the map $j^* : \mathrm{K}^*_T(X_1) \longrightarrow \mathrm{K}^*_T(X^T)$.*

The Chang-Skjelbred Theorem [10] is Theorem 9.3 for Borel's equivariant cohomology. The corresponding theorem for delocalized equivariant cohomology is an immediate corollary of Theorem 9.3.

## Appendix. The circle action on the sphere

According to Guillemin and Sternberg [12, §11.7], whenever a torus acts on a surface with non-empty fixed point set, the surface is diffeomorphic to the sphere and action is effectively the rotation of the sphere around the $z$-axis. Their subsequent computation of the equivariant cohomology makes use of equivariant formality and we now show that it is straightforward to carry out this computation, even for non-commutative groups, by resolving the sphere.

Thus consider a compact group $G$ (not necessarily Abelian) acting smoothly on $X = \mathbb{S}^2$. Suppose that $G$ has a codimension one normal subgroup $K$ that acts trivially on $X$, and that the quotient $\mathbb{S}^1 = G/K$ acts on $X$ by rotating around the $z$-axis (in the usual embedding of $X$ into $\mathbb{R}^3$).

Thus the $G$-action has two isotropy types: the 'north and south poles', $\{N, S\}$ constitute an isotropy type corresponding to $G$, and their complement constitutes an isotropy type corresponding to $K$. This action is resolved by lifting to

$$Y(X) = [X; \{N, S\}].$$

The boundary of $Y(X)$ is the disjoint union of two circles and the boundary fibration is the map from each circle to the corresponding pole. In this case

$$\mathcal{C}^\infty_{G,\Phi}(Y(X);\Lambda^*) = \\ \{(\omega, f_N, f_S) \in \mathcal{C}^\infty_G(Y; S(\mathfrak{g}^*)\otimes\Lambda^*) \oplus \bigoplus_{N,S} S(\mathfrak{g}^*)^G ; i_N^*\omega = f_N, i_S^*\omega = f_S\}$$

where we are identifying $f_N$ with $f_N \otimes 1 \in \mathcal{C}^\infty_G(\mathbb{S}^1; S(\mathfrak{g}^*)\otimes\Lambda^*)$ and similarly with $f_S$.

We can identify the orbit space $Z(X) = G\backslash Y(X)$ with the unit interval, and in this case the Borel bundle is the trivial $S(\mathfrak{k}^*)^K$ bundle. Thus the reduced Cartan complex is

$$\mathcal{C}^\infty_\Phi([0,1]; B^*\otimes\Lambda^*) = \Big\{(\omega, f_N, f_S) \in \big(S(\mathfrak{k}^*)^K \otimes \mathcal{C}^\infty([0,1];\Lambda^*)\big) \oplus \bigoplus_{N,S} S(\mathfrak{g}^*)^G; \\ i_0^*\omega = r(f_N), i_1^*\omega = r(f_S)\Big\}$$

where $r : S(\mathfrak{g}^*)^G \longrightarrow S(\mathfrak{k}^*)^K$ is the natural restriction map, and the differential is the exterior derivative on the first factor. Since the interval is contractible, we find

$$\begin{aligned} \mathrm{H}^*_G(\mathbb{S}^2) &= \mathrm{H}^*(\mathcal{C}^\infty_{G,\Phi}(Y(X); S(\mathfrak{g}^*)\otimes\Lambda^*), d_{\mathrm{eq}}) = \mathrm{H}^*(\mathcal{C}^\infty_\Phi([0,1]; B^*\otimes\Lambda^*), d) \\ &= \{(f_N, f_S) \in S(\mathfrak{g}^*)^G \oplus S(\mathfrak{g}^*)^G ; r(f_N) = r(f_S)\} \end{aligned}$$

and so $\mathrm{H}^q_G(\mathbb{S}^2)$ is trivial if $q$ is odd and is non-trivial for all even $q \geq 0$.

The representation bundle is also trivial in this case, so the same reasoning shows that

$$\mathrm{K}^0_G(\mathbb{S}^2) = \{(\tau_N, \tau_S) \in \mathcal{R}(G) \oplus \mathcal{R}(G) : \rho(\tau_N) = \rho(\tau_S)\}, \quad \mathrm{K}^1_G(\mathbb{S}^2) = 0$$

where $\rho : \mathcal{R}(G) \longrightarrow \mathcal{R}(K)$ is the restriction map. Indeed, classes in $\mathrm{K}_\Phi([0,1];\mathcal{R})$ consist of vector bundles over the interval and its end points with, respectively, coefficients in $\mathcal{R}(K)$ and $\mathcal{R}(G)$ and the compatibility condition is induced by the restriction map.

Finally note that the complex (8.1) in this case is given by

$$\mathcal{C}^\infty_\Phi([0,1]; \mathcal{R}\otimes\Lambda^*) = \\ \{(\omega, \tau_N, \tau_S) \in (\mathcal{R}(K)\otimes\mathcal{C}^\infty([0,1];\Lambda^*)) \oplus \bigoplus_{N,S} \mathcal{R}(G); i_0^*\omega = \rho(\tau_N), i_1^*\omega = \rho(\tau_S)\}$$

with differential given by the exterior derivative on the first factor. Thus the delocalized equivariant cohomology is

$$\mathrm{H}^{\mathrm{even}}_{G,\mathrm{dl}}(X) = \{(\tau_N, \tau_S) \in \mathcal{R}(G)\oplus\mathcal{R}(G); \rho(\tau_N) = \rho(\tau_S)\},\ \mathrm{H}^{\mathrm{odd}}_{G,\mathrm{dl}}(X) = 0.$$

The Chern character from equivariant K-theory to delocalized equivariant cohomology is the identity, while the Chern character into (localized) equivariant cohomology is localization at the identity in $G$.

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